CONTINUOUS ORBIT EQUIVALENCE RIGIDITY

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Abstract. We take the first steps towards a better understanding of continuous orbit equivalence, i.e., topological orbit equivalence with continuous cocycles. First, we characterise continuous orbit equivalence in terms of isomorphisms of C*-crossed products preserving Cartan subalgebras. This is the topological analogue of the classical result by Singer and Feldman-Moore in the measurable setting. Secondly, we turn to continuous orbit equivalence rigidity, i.e., the question whether for certain classes of topological dynamical systems, continuous orbit equivalence implies conjugacy. We show that this is not always the case by constructing topological dynamical systems (actions of free abelian groups, and also non-abelian free groups) which are continuously orbit equivalent but not conjugate. Furthermore, we prove positive rigidity results. For instance, for solvable duality groups, general topological Bernoulli actions and certain subshifts of full shifts over finite alphabets are rigid.

1. Introduction

From its very beginning on, the theory of operator algebras has been closely related to ergodic theory and dynamical systems. The bridge between these subjects is built by crossed product constructions, attaching von Neumann algebras to measure-preserving dynamical systems and C*-algebras to topological dynamical systems. In the setting of von Neumann algebras, the crossed product construction, also called group-measure space construction, played an important role in the classification of injective factors. Similarly, in the C*-algebraic setting, crossed products attached to topological dynamical systems provide interesting examples which are challenging to classify and lead to new insights.

If we want to further develop the relationship between operator algebras and dynamical systems, we are led to the following crucial and natural question:

How much information do these crossed product constructions contain about the underlying dynamical systems?

It turns out that the crossed product itself might contain very little information, but if we consider the crossed product together with a canonical commutative subalgebra, then our question can be answered in a systematic way.

To explain this in the measurable and von Neumann algebraic setting, let $G \curvearrowright X$ and $H \curvearrowright Y$ be probability measure preserving actions. Here, our measure spaces are standard, our groups are discrete and countable, and they act by Borel automorphisms. We say that $G \curvearrowright X$ and $H \curvearrowright Y$ are orbit equivalent if there exists an isomorphism of measure spaces $\varphi: X \to Y$ with $\varphi(G.x) = H.\varphi(x)$ for a.e. $x \in X$. Here is a classical result (see [31, 11, 32] for more details):

2010 *Mathematics Subject Classification.* Primary 37B05; Secondary 37A20, 46L05.

Research supported by EPSRC grant EP/M009718/1.

Theorem 1.1 ([29, 8, 9]). *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be (essentially) free probability measure preserving actions. Then $G \curvearrowright X$ and $H \curvearrowright Y$ are orbit equivalent if and only if there is a vN-isomorphism $\Phi : L^\infty(X) \rtimes G \overset{\cong}{\longrightarrow} L^\infty(Y) \rtimes H$ with $\Phi(L^\infty(X)) = L^\infty(Y)$.*

Our first result carries over Theorem 1.1 to the topological setting. Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topological dynamical systems. This means that G and H are countable discrete groups acting by homeomorphisms on locally compact second countable Hausdorff spaces X and Y.

We say that $G \curvearrowright X$ and $H \curvearrowright Y$ are continuously orbit equivalent if there exists a homeomorphism $\varphi : X \overset{\cong}{\longrightarrow} Y$ with inverse $\psi = \varphi^{-1} : Y \overset{\cong}{\longrightarrow} X$ and continuous maps $a : G \times X \to H$, $b : H \times Y \to G$ such that $\varphi(g.x) = a(g,x).\varphi(x)$ and $\psi(h.y) = b(h,y).\psi(y)$ for all $g \in G$, $x \in X$, $h \in H$ and $y \in Y$. Note that G and H carry the discrete topology. This notion of continuous orbit equivalence has been studied in special cases (see [14, 2, 30]), but not in the general setting. There is also a weaker notion of topological orbit equivalence which has been studied intensively for $\mathbb{Z}^n$-actions on the Cantor set in the remarkable papers [14, 12, 13].

Moreover, let $G \ltimes X$ and $H \ltimes Y$ be the transformation groupoids attached to $G \curvearrowright X$ and $H \curvearrowright Y$. Here is the topological analogue of Theorem 1.1:

Theorem 1.2. *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free systems. The following are equivalent:*

- *$G \curvearrowright X$ and $H \curvearrowright Y$ are continuously orbit equivalent;*
- *$G \ltimes X$ and $H \ltimes Y$ are isomorphic as topological groupoids;*
- *there is a C*-isomorphism $\Phi : C_0(X) \rtimes_r G \overset{\cong}{\longrightarrow} C_0(Y) \rtimes_r H$ with $\Phi(C_0(X)) = C_0(Y)$.*

The equivalence of the second and third item is due to J. Renault (see [24, Proposition 4.13]).

Here and in the sequel, "topologically free system" stands for "topologically free topological dynamical system". Theorem 1.2 tells us that – at least for our purposes – continuous orbit equivalence is a good topological analogue of orbit equivalence in the measurable setting.

As an immediate consequence, we obtain for connected spaces

Corollary 1.3. *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free systems on connected spaces X and Y. Then Let $G \curvearrowright X$ and $H \curvearrowright Y$ are conjugate if there is a C*-isomorphism $\Phi : C_0(X) \rtimes_r G \overset{\cong}{\longrightarrow} C_0(Y) \rtimes_r H$ with $\Phi(C_0(X)) = C_0(Y)$.*

Here, $G \curvearrowright X$ and $H \curvearrowright Y$ are conjugate if there is a homeomorphism $\varphi : X \overset{\cong}{\longrightarrow} Y$ and a group isomorphism $\rho : G \overset{\cong}{\longrightarrow} H$ such that $\varphi(g.x) = \rho(g).\varphi(x)$ for all $g \in G$ and $x \in X$.

In the measurable setting, orbit equivalence rigidity has become a well-established key notion. The idea is to find classes of actions for which orbit equivalence already implies conjugacy. Indeed, impressive orbit equivalence rigidity results have been obtained in [33, 10, 21, 22, 23, 16, 15]. Viewing continuous orbit equivalence as the topological analogue of orbit equivalence, a natural question is whether there are rigidity phenomena for continuous orbit equivalence.

The only result known in this context is due to [2], which says that if $\mathbb{Z} \curvearrowright X$ and $\mathbb{Z} \curvearrowright Y$ are topologically transitive, topologically free systems on compact spaces X and Y, then $\mathbb{Z} \curvearrowright X$ and $\mathbb{Z} \curvearrowright Y$ must already be conjugate if they are continuously orbit equivalent. Apart from this, not much else seems to be known about continuous orbit equivalence rigidity.

The main goal of the present paper is to take the first steps towards a better understanding of continuous orbit equivalence rigidity.

First of all, we construct examples of topological dynamical systems which are continuously orbit equivalent but not conjugate. This ensures that the comparison between continuous orbit equivalence and conjugacy is really interesting. A first class of examples is given by products of odometer actions. A second family of examples is constructed from boundary actions of non-abelian free groups and odometer actions, inspired by [30].

Secondly, we prove positive results in continuous orbit equivalence rigidity. We need the notion of projective dimension for modules and cohomological dimension for groups. Given a module N over a ring R, its projective dimension $\mathrm{pd}_R(N)$ over R is the smallest length of a projective resolution of N over R. Given a group G, the cohomological dimension of G is defined as $\mathrm{cd}(G) := \mathrm{pd}_{\mathbb{Z}G}(\mathbb{Z})$, where $\mathbb{Z}$ is viewed as a trivial $\mathbb{Z}G$-module.

Theorem 1.4. *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free systems. Assume that X is compact and that $C(X,\mathbb{Z}) = \mathbb{Z} \cdot 1 \oplus N$ as $\mathbb{Z}G$-modules, with $\mathrm{pd}_{\mathbb{Z}G}(N) < \mathrm{cd}(G) - 1$. Furthermore, let G be a duality group in the sense of* [3, Chapter VIII, § 10] *and H a solvable group. Then $G \curvearrowright X$ and $H \curvearrowright Y$ must be conjugate if they are continuously orbit equivalent.*

For instance, every torsion-free polycyclic group is a duality group.

Let us call a topologically free system $G \curvearrowright X$ continuous orbit equivalence rigid if for every topologically free system $H \curvearrowright Y$, $G \curvearrowright X$ and $H \curvearrowright Y$ must be conjugate if they are continuously orbit equivalent. We now present explicit examples of topologically free systems which satisfy the conditions of Theorem 1.4.

Theorem 1.5. *The following topologically free systems are continuous orbit equivalence rigid:*

- *Topological Bernoulli actions $G \curvearrowright X_0^G$, where X_0 is a compact space with $|X_0| > 1$ and G is a solvable duality group;*
- *topologically free subshifts of $G \curvearrowright A^G$ whose forbidden words avoid a fixed letter, where $|A| < \infty$ and G is a solvable duality group, for instance*
  - *the golden mean: the subshift consisting of all $x \in \{0,1\}^{\mathbb{Z}^d}$ in which the 1's are isolated* [26, Example 10.7],
  - *the iceberg model: the subshift consisting of all $x \in \{-M,\dots,M\}^{\mathbb{Z}^d}$ in which no positive integer is adjacent to a negative one* [26, Example 10.8],
  - *the hard core model: the subshift consisting of all $x \in \{0,\dots,m\}^{\mathbb{Z}^d}$ in which no two positive integers are next to each other* [4, Example 2].

Remark 1.6. Building on work of K. Schmidt on cocycle rigidity [25, 26], we obtain more examples of continuous orbit equivalence rigid systems, namely

- chessboards: the subshift consisting of all $x \in \{1,\dots,n\}^{\mathbb{Z}^2}$ in which no number is adjacent to the same one [26, Example 10.6],
- long dominoes: the Wang shift consisting of all tilings of $\mathbb{R}^2$ by integer translates of "long dominoes" of the form $[0,m] \times [0,1]$ or $[0,1] \times [0,n]$ [26, Example 10.9],

- dominoes in three dimensions: the Wang shift consisting of all tilings of $\mathbb{R}^3$ by integer translates of "three-dimensional dominoes" of the form $[0,2] \times [0,1] \times [0,1]$, $[0,1] \times [0,2] \times [0,1]$ or $[0,1] \times [0,1] \times [0,2]$ [26, Example 10.10].

The proofs of these rigidity results consist of three main ingredients, which are interesting in their own right. The first ingredient establishes a link between continuous cocycle rigidity and continuous orbit equivalence rigidity. Let $G \curvearrowright X$ be a topological dynamical system, and let H be a group. A continuous map $a: G \times X \to H$ is called a continuous H-cocycle for $G \curvearrowright X$ if $a(g_1 g_2, x) = a(g_1, g_2.x) a(g_2, x)$ for all $g_1, g_2 \in G$ and $x \in X$. In particular, we can view any group homomorphism $\rho: G \to H$ as a cocycle given by $(g, x) \mapsto \rho(g)$. Continuous cocycles a and a' are called cohomologous if there exists a continuous map $u: X \to H$ such that $a(g, x) = u(g.x) a'(g, x) u(x)^{-1}$. We say that $G \curvearrowright X$ is continuous H-cocycle rigid if every continuous H-cocycle for $G \curvearrowright X$ is cohomologous to some group homomorphism $\rho: G \to H$.

Theorem 1.7. *Let G be a torsion-free amenable group. Assume that $G \curvearrowright X$ and $H \curvearrowright Y$ are topologically free systems on compact spaces X and Y, and that $G \curvearrowright X$ and $H \curvearrowright Y$ are continuously orbit equivalent. If $G \curvearrowright X$ is continuous H-cocycle rigid, then $G \curvearrowright X$ and $H \curvearrowright Y$ must be conjugate.*

The second ingredient establishes continuous cocycle rigidity for certain dynamical systems. For the proof, we use completely different methods than in [25, 26].

Theorem 1.8. *Let G be a duality group. Let X be a compact space, and suppose that $G \curvearrowright X$ is a topological dynamical system such that $C(X, \mathbb{Z}) = \mathbb{Z} \cdot 1 \oplus N$ as $\mathbb{Z}G$-modules, with $\mathrm{pd}_{\mathbb{Z}G}(N) < \mathrm{cd}(G) - 1$. Then $G \curvearrowright X$ is continuous H-cocycle rigid for every solvable group H.*

Using different methods, N.-P. Chung and Y. Jiang have established rigidity results for continuous cocycles in [5].

The third and final ingredient is a connection between continuous orbit equivalence and quasi-isometries. For our purposes, the following result suffices:

Theorem 1.9. *Let topologically free systems $G \curvearrowright X$, $H \curvearrowright Y$ on compact spaces X, Y be continuously orbit equivalent. If one of the groups (G or H) is finitely generated, then G and H must be quasi-isometric.*

This is a special case of the results in [18]. Independently, K. Medynets, R. Sauer and A. Thom obtained a strengthening of Theorem 1.9 in [20].

Just to give a glimpse on possible future research directions connected with the present piece of work, we mention the following interesting question:

Are topological Bernoulli actions $G \curvearrowright X_0^G$ for arbitrary torsion-free groups always continuously orbit equivalence rigid?

It would also be very interesting to study the notion of Cartan subalgebras of C*-algebras [24] (see also Remark 2.4). There is work in progress to do so in general, and – jointly with S. Barlak – in relation with the UCT problem [1]. Furthermore, Theorem 1.2 can be generalized to partial systems [17]. This can be used to generalize and conceptually explain recent results [19] on orbit equivalence and Cartan isomorphism for shifts of finite type.

In § 2, we introduce the notion of continuous orbit equivalence, make some general observations and prove Theorem 1.2. Moreover, we discuss known examples for continuous orbit equivalence rigidity and construct counterexamples for which continuous orbit equivalence does not imply conjugacy in § 3. In § 4, we introduce the notion of continuous cocycle rigidity, study the connection to continuous orbit equivalence rigidity, and prove Theorem 1.7. In § 5, we study continuous cocycle rigidity using non-abelian group cohomology, discuss particular examples, and prove Theorem 1.8. Finally, in § 6, we prove Theorem 1.4 and Theorem 1.5.

I thank David Kerr for inspiring discussions about continuous orbit equivalence, and Andreas Thom for informing me about [20]. I would also like to thank the referee for careful proofreading and for many helpful comments and suggestions which improved this paper.

2. Continuous orbit equivalence, transformation groupoids and Cartan pairs

All our groups are discrete and countable, and all our topological spaces are locally compact, second countable and Hausdorff. By a topological dynamical system, we mean an action of a group on a topological space by homeomorphisms.

Let $G \curvearrowright X$ be a topological dynamical system. The G-action is denoted by $G \times X \to X$, $(g,x) \mapsto g.x$. For $x \in X$, let $G_x = \{g \in G \colon g.x = x\}$ be its stabilizer group. The transformation groupoid $G \ltimes X$ attached to $G \curvearrowright X$ is given by the set $G \times X$ with multiplication $(g',x')(g,x) = (g'g,x)$ if $x' = g.x$, inversion $(g,x)^{-1} = (g^{-1}, g.x)$, range map $r(g,x) = g.x$ and source map $s(g,x) = x$. Obviously, $G \ltimes X$ is étale. The reduced groupoid C*-algebra $C_r^*(G \ltimes X)$ is canonically isomorphic to $C_0(X) \rtimes_r G$. Moreover, we have a canonical embedding $C_0(X) \hookrightarrow C_0(X) \rtimes_r G$.

Definition 2.1. *$G \curvearrowright X$ is called topologically free if for every $e \neq g \in G$, $\{x \in X \colon g.x \neq x\}$ is dense in X.*

From now on, for the sake of brevity, we write "topologically free system" for "topologically free topological dynamical system".

Lemma 2.2. *$G \curvearrowright X$ is topologically free if and only if $\{x \in X \colon G_x = \{e\}\}$ is dense in X.*

Proof. "$\Leftarrow$" is clear. For "$\Rightarrow$", note that by topological freeness, $\{x \in X \colon g.x \neq x\}$ is dense (and open) in X for all $e \neq g \in G$. Thus $\{x \in X \colon G_x = \{e\}\} = \bigcap_{e \neq g \in G} \{x \in X \colon g.x \neq x\}$ must be dense in X by the Baire category theorem. □

Corollary 2.3. *$G \curvearrowright X$ is topologically free if and only if the transformation groupoid $G \ltimes X$ is topologically principal.*

Proof. By definition (see [24]), $G \ltimes X$ is topologically principal if and only if the set of points in X with trivial isotropy is dense in X. But this set coincides with $\{x \in X \colon G_x = \{e\}\}$. Thus Lemma 2.2 implies our corollary. $\square$

Remark 2.4. Corollary 2.3 shows that if $G \curvearrowright X$ is topologically free, then the pair $(C_0(X) \rtimes_r G, C_0(X))$ is a Cartan pair in the sense of [24, Definition 5.1].

Recall the following definition from the introduction:

Definition 2.5. *Topological dynamical systems $G \curvearrowright X$ and $H \curvearrowright Y$ are continuously orbit equivalent (we write $G \curvearrowright X \sim_{coe} H \curvearrowright Y$) if there exists a homeomorphism $\varphi : X \xrightarrow{\cong} Y$ with inverse $\psi = \varphi^{-1}$ and continuous maps $a : G \times X \to H$, $b : H \times Y \to G$ such that*

$$\varphi(g.x) = a(g,x).\varphi(x) \tag{1}$$

$$\psi(h.y) = b(h,y).\psi(y) \tag{2}$$

for all $g \in G$, $x \in X$, $h \in H$ and $y \in Y$.

Remark 2.6. (1) implies $\varphi(G.x) \subseteq H.\varphi(x)$ for all $x \in X$, and (2) implies $\psi(H.y) \subseteq G.\psi(y)$ for all $y \in Y$. Thus, $\varphi(G.x) = H.\varphi(x)$ and $\psi(H.y) = G.\psi(y)$.

Remark 2.7. If $H \curvearrowright Y$ is topologically free, then a is uniquely determined by (1), and by symmetry, if $G \curvearrowright X$ is topologically free, then b is uniquely determined by (2). The reason is as follows: Suppose that $a' : G \times X \to H$ is another continuous map with $\varphi(g.x) = a'(g,x).\varphi(x)$. For arbitrary $g \in G$ and $x \in X$, there exists an open neighbourhood U of x such that a and a' are constant on $\{g\} \times U$, with values h and h' in H, say. Then for every $\bar{x} \in U$, $\varphi(g.\bar{x}) = h.\varphi(\bar{x}) = h'.\varphi(\bar{x})$. Topological freeness implies $h = h'$, in particular $a(g,x) = a'(g,x)$.

Lemma 2.8. *In Definition 2.5, if $H \curvearrowright Y$ is topologically free, then*

$$a(g_1 g_2, x) = a(g_1, g_2.x) a(g_2, x)$$

for all $g_1, g_2 \in G$ and $x \in X$.

Proof. Let $g_1, g_2 \in G$ and $x \in X$ be arbitrary. Choose an open neighbourhood U of $x \in X$ such that $a(g_1 g_2, \bar{x}) = a(g_1 g_2, x)$, $a(g_1, g_2.\bar{x}) = a(g_1, g_2.x)$ and $a(g_2, \bar{x}) = a(g_2, x)$ for all $\bar{x} \in U$. Then for all $\bar{x} \in U$, $\varphi(g_1 g_2.\bar{x}) = \varphi(g_1.(g_2.\bar{x})) = a(g_1, g_2.\bar{x}).\varphi(g_2.\bar{x}) = a(g_1, g_2.\bar{x}) a(g_2, \bar{x}).\varphi(\bar{x}) = a(g_1, g_2.x) a(g_2, x).\varphi(\bar{x})$, but also $\varphi(g_1 g_2.\bar{x}) = a(g_1 g_2, \bar{x}).\varphi(\bar{x}) = a(g_1 g_2, x).\varphi(\bar{x})$. By topological freeness, $a(g_1 g_2, x) = a(g_1, g_2.x) a(g_2, x)$. $\square$

Lemma 2.9. *In the situation of Definition 2.5, let $Y_f = \{y \in Y \colon H_y = \{e\}\}$. For every $x \in \psi(Y_f)$, $a_x : G \to H$, $g \mapsto a(g,x)$ is bijective.*

Proof. Since $\varphi(x) \in Y_f$, a_x is injective. To prove surjectivity, take $h \in H$. Since by Remark 2.6, $\varphi(G.x) = H.\varphi(x)$, there exists $g \in G$ with $h.\varphi(x) = \varphi(g.x) = a(g,x).\varphi(x)$. As $\varphi(x) \in Y_f$, we conclude that $h = a(g,x) = a_x(g)$. $\square$

Lemma 2.10. *In the situation of Definition 2.5, assume that $G \curvearrowright X$ and $H \curvearrowright Y$ are topologically free. Then*

$$b(a(g,x), \varphi(x)) = g \text{ for all } g \in G,\ x \in X, \tag{3}$$

and b is uniquely determined by (3).

Proof. Let $h := a(g,x)$. Then $\varphi(g.x) = h.\varphi(x)$, so $g.x = \psi(h.\varphi(x)) = b(h,\varphi(x)).x$. Since this equation holds in an open neighbourhood of x, topological freeness implies $b(a(g,x),\varphi(x)) = g$. Moreover, note that for all $x \in \psi(Y_f)$, $a_x(G) = H$ by Lemma 2.9. Hence (3) determines b on $H \times Y_f$. But since Y_f is dense in Y by topological freeness, and because b is continuous, (3) determines b on $H \times Y$. $\square$

Corollary 2.11. *In the situation of Definition 2.5, assume that $G \curvearrowright X$ and $H \curvearrowright Y$ are topologically free. Let $X_f = \{x \in X\colon G_x = \{e\}\}$ and $Y_f = \{y \in Y\colon H_y = \{e\}\}$. Then $\varphi(X_f) = Y_f$. In particular, for every $x \in X$ with $G_x = \{e\}$, $a_x : G \to H$, $g \mapsto a(g,x)$ is bijective.*

Proof. By symmetry, we just have to show $\varphi(X_f) \subseteq Y_f$. Take $x \in X_f$, and let $y = \varphi(x)$. Suppose that $h \in H$ satisfies $h.y = y$. Then $x = \psi(y) = \psi(h.y) = b(h,y).\psi(y) = b(h,y).x$, and therefore $b(h,y) = e$ since $x \in X_f$. But by the analogue of (3) with reversed roles for a and b, we get $e = a(e,x) = a(b(h,y),x) = h$. Hence $y \in Y_f$. $\square$

We are now ready for the proof of Theorem 1.2.

Theorem (Theorem 1.2)**.** *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free systems. The following are equivalent:*

(i) $G \curvearrowright X \ \sim_{coe}\ H \curvearrowright Y$;
(ii) $G \ltimes X \cong H \ltimes Y$ *(as topological groupoids);*
(iii) *there is a C^*-isomorphism* $\Phi : C_0(X) \rtimes_r G \xrightarrow{\cong} C_0(Y) \rtimes_r H$ *with* $\Phi(C_0(X)) = C_0(Y)$.

Proof. (i) $\Rightarrow$ (ii): Assume that $G \curvearrowright X \ \sim_{\text{coe}}\ H \curvearrowright Y$, and let φ, ψ, a and b be as in Definition 2.5. Then $G \ltimes X \to H \ltimes Y$, $(g,x) \mapsto (a(g,x),\varphi(x))$ and $H \ltimes Y \to G \ltimes X$, $(h,y) \mapsto (b(h,y),\psi(y))$ are certainly continuous groupoid morphisms, and they are inverse to each other due to (3) and the analogue of (3) with reversed roles for a and b.

(ii) $\Rightarrow$ (i): Let $\chi : G \ltimes X \xrightarrow{\cong} H \ltimes Y$ be an isomorphism of topological groupoids. Set $\varphi = \chi|_X : X \xrightarrow{\cong} Y$ and let a be the composition $G \ltimes X \xrightarrow{\chi} H \ltimes Y \to H$, where the second map is $H \ltimes Y \to H, (h,y) \mapsto h$. Then a is obviously continuous, and $\varphi(g.x) = \chi(r(g,x)) = r(\chi(g,x)) = r(a(g,x),\varphi(x)) = a(g,x).\varphi(x)$. Similarly, for $\psi = \varphi^{-1}$, if we let b be the composition $H \ltimes Y \xrightarrow{\chi^{-1}} G \ltimes X \to G$, where the second map is $G \ltimes X \to G, (g,x) \mapsto g$, then $\psi(h.y) = b(h,y).\psi(y)$.

(ii) $\Leftrightarrow$ (iii) is [24, Proposition 4.13], where we have to use Corollary 2.3. $\square$

Corollary 2.12. *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free systems, and assume that $G \curvearrowright X \ \sim_{coe}\ H \curvearrowright Y$. If $G \curvearrowright X$ has a global fixed point (a point $x \in X$ with $G_x = G$), then so does $H \curvearrowright Y$, and we must have $G \cong H$.*

Proof. This is obvious as $G \curvearrowright X \ \sim_{\text{coe}}\ H \curvearrowright Y$ implies $G \ltimes X \cong H \ltimes Y$ by Theorem 1.2. $\square$

3. Continuous orbit equivalence rigidity: Examples and counterexamples

Let us compare continuous orbit equivalence with conjugacy.

Definition 3.1. *Topological dynamical systems $G \curvearrowright X$ and $H \curvearrowright Y$ are conjugate (we write $G \curvearrowright X \ \sim_{conj}\ H \curvearrowright Y$) if there is a homeomorphism $\varphi : X \xrightarrow{\cong} Y$ and a group isomorphism $\rho : G \xrightarrow{\cong} H$ such that for every $g \in G$ and $x \in X$, $\varphi(g.x) = \rho(g).\varphi(x)$.*

Obviously, $G \curvearrowright X \ \sim_{\text{conj}} \ H \curvearrowright Y$ implies $G \curvearrowright X \ \sim_{\text{coe}} \ H \curvearrowright Y$. Are there classes of dynamical systems where the converse holds, where continuous orbit equivalence implies conjugacy?

Here is a first class of examples, for which continuous orbit equivalence rigidity holds: Suppose that $G \curvearrowright X$ is a topologically free system on a connected space X. If $G \curvearrowright X \ \sim_{\text{coe}} \ H \curvearrowright Y$ for some topologically free system $H \curvearrowright Y$, then $G \curvearrowright X \ \sim_{\text{conj}} \ H \curvearrowright Y$. The reason is that the function a in Definition 2.5 is continuous, hence for every $g \in G$, $a|_{\{g\} \times X}$ is constant because X is connected and H is discrete. Hence $a(g,x) = \rho(g)$ for some map $\rho : G \to H$, and ρ has to be a homomorphism (by Lemma 2.8) and bijective (by Lemmma 2.9). This proves Corollary 1.3.

This observation means that if we focus on discrete groups, it is natural to restrict our discussion to topological dynamical systems on totally disconnected spaces.

Here is another result in continuous orbit equivalence rigidity:

Theorem 3.2 ([2, Theorem 3.2]). *Let $\mathbb{Z} \curvearrowright X$ and $\mathbb{Z} \curvearrowright Y$ be topologically free systems on compact spaces X and Y. Assume that $\mathbb{Z} \curvearrowright X$ is topologically transitive.*

If $\mathbb{Z} \curvearrowright X \ \sim_{coe} \ \mathbb{Z} \curvearrowright Y$, then $\mathbb{Z} \curvearrowright X \ \sim_{conj} \ \mathbb{Z} \curvearrowright Y$.

In this theorem, while the groups are fixed, the assumptions on the actions are very mild. Therefore, an immediate question is whether there are counterexamples to continuous orbit equivalence rigidity at all, i.e., examples of topological dynamical systems which are continuously orbit equivalent but not conjugate.

3.1. **Products of odometer transformations.** For every natural number $m \geq 1$, there are uniquely determined $v_p(m) \in \{0,1,2,\dots\}$ such that $m = \prod_p p^{v_p(m)}$, where p runs through all prime numbers. Similary, for every supernatural number M, there are uniquely determined $v_p(M) \in \{0,1,2,\dots\} \cup \{\infty\}$ with $\sum_p v_p(M) = \infty$ such that $M = \prod_p p^{v_p(M)}$. The only difference is that for a natural number m, we have $\sum_p v_p(m) < \infty$, while for a supernatural number M, we have $\sum_p v_p(M) = \infty$.

The odometer action $\mathbb{Z} \curvearrowright \mathbb{Z}/M$ corresponding to the supernatural number M is constructed as follows: Choose a sequence $(m_k)_k$ of natural numbers such that, for all primes p, $v_p(m_k) \nearrow v_p(M)$ for $k \to \infty$. Then set $\mathbb{Z}/M = \varprojlim_k \mathbb{Z}/m_k$. The canonical projections $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/m_k$ induce a group embedding $\mathbb{Z} \hookrightarrow \mathbb{Z}/M$, and this in turn yields an action $\mathbb{Z} \curvearrowright \mathbb{Z}/M$ which we call the odometer transformation for M.

Theorem 3.3. *For supernatural numbers $M_1, \dots, M_r$ and $N_1, \dots, N_s$, the following are equivalent:*

(i) $\mathbb{Z}^r \curvearrowright \prod_{i=1}^r \mathbb{Z}/M_i \ \sim_{coe} \ \mathbb{Z}^s \curvearrowright \prod_{j=1}^s \mathbb{Z}/N_j$;
(ii) $C_0(\prod_{i=1}^r \mathbb{Z}/M_i) \rtimes \mathbb{Z}^r \cong C_0(\prod_{j=1}^s \mathbb{Z}/N_j) \rtimes \mathbb{Z}^s$;
(iii) $(K_*(C_0(\prod_{i=1}^r \mathbb{Z}/M_i) \rtimes \mathbb{Z}^r), [1]_0) \cong (K_*(C_0(\prod_{j=1}^s \mathbb{Z}/N_j) \rtimes \mathbb{Z}^s), [1]_0)$;
(iv) *$r = s$, there exists $\sigma \in S_r$, natural numbers $m_1, \dots, m_r$ and $n_1, \dots, n_r$ such that for all $1 \leq i \leq r$, $m_i M_i = n_{\sigma(i)} N_{\sigma(i)}$, and $\prod_{i=1}^r M_i = \prod_{j=1}^r N_j$.*

Proof. (i) $\Rightarrow$ (ii) follows from Theorem 1.2 as our systems are free.

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (iv): K_* stands for $K_0 \oplus K_1$. Clearly, $(K_0(C(\mathbb{Z}/M) \rtimes \mathbb{Z}), [1]_0) \cong (\mathbb{Z}[M^{-1}], 1)$ and $K_1(C(\mathbb{Z}/M) \rtimes \mathbb{Z}) \cong \mathbb{Z}$. Here $\mathbb{Z}[M^{-1}] = \left\{ \frac{x}{m} \in \mathbb{Q} \colon m \mid M \right\}$. So $K_*(C(\prod_{i=1}^r \mathbb{Z}/M_i) \rtimes \mathbb{Z}^r) \cong \bigoplus_{I \subseteq \{1,\dots,r\}} \mathbb{Z}[(\prod_{i \in I} M_i)^{-1}]$ and $[1]_0$ corresponds to $1 \in \mathbb{Z}[(\prod_{i=1}^r M_i)^{-1}]$ $(I = \{1,\dots,r\})$. Therefore, $\mathbb{Q}^{2^r} \cong K_*(C(\prod_{i=1}^r \mathbb{Z}/M_i) \rtimes \mathbb{Z}^r) \otimes \mathbb{Q} \cong K_*(C(\prod_{j=1}^s \mathbb{Z}/N_j) \rtimes \mathbb{Z}^s) \otimes \mathbb{Q} \cong \mathbb{Q}^{2^s}$, and this implies $r = s$. Moreover, as a K_*-isomorphism preserves $[1]_0$, it restricts to an isomorphism $\mathbb{Z}[(\prod_{i=1}^r M_i)^{-1}] \cong \mathbb{Z}[(\prod_{j=1}^r N_j)^{-1}]$ sending 1 to 1. This implies $\prod_{i=1}^r M_i = \prod_{j=1}^r N_j$.
Given supernatural numbers M and N, we define $M \lesssim N$ if there exists $n \in \mathbb{N}$ with $M \mid nN$ $(v_p(M) \leq v_p(nN))$. We define $M \sim N$ if $M \lesssim N$ and $N \lesssim M$. It is immediate that there exists a non-zero homomorphism $\mathbb{Z}[M^{-1}] \to \mathbb{Z}[N^{-1}]$ if and only if $M \lesssim N$. Set $\mathcal{M} = \{M_i \colon 1 \leq i \leq r\}$, $\bigwedge \mathcal{M} = \left\{ \prod_{i \in I} M_i \colon I \subseteq \{1,\dots,r\} \right\}$ and $\mathcal{N} = \{N_j \colon 1 \leq j \leq r\}$, $\bigwedge \mathcal{N} = \left\{ \prod_{j \in J} N_j \colon J \subseteq \{1,\dots,r\} \right\}$. Using the assumption that $\bigoplus_{I \subseteq \{1,\dots,r\}} \mathbb{Z}[(\prod_{i \in I} M_i)^{-1}] \cong \bigoplus_{J \subseteq \{1,\dots,r\}} \mathbb{Z}[(\prod_{j \in J} N_j)^{-1}]$, a straightforward inductive argument shows that for every equivalence class $\mathcal{S}$ of supernatural numbers with respect to $\sim$, $|\mathcal{S} \cap \bigwedge \mathcal{M}| = |\mathcal{S} \cap \bigwedge \mathcal{N}|$, and then also $|\mathcal{S} \cap \mathcal{M}| = |\mathcal{S} \cap \mathcal{N}|$.

(iv) $\Rightarrow$ (i): We need the following observation: Let l be a natural number and $\lambda_l : \mathbb{Z}/l \curvearrowright \mathbb{Z}/l$ the canonical action. Let L be a supernatural number, $X = \mathbb{Z}/lL$, $\tilde{X} = l \cdot (\mathbb{Z}/lL)$, $\alpha_{lL} : \mathbb{Z} \curvearrowright X$ the odometer transformation for lL, and $\tilde{\alpha} = \alpha|_{L\mathbb{Z}} : l\mathbb{Z} \curvearrowright \tilde{X}$. We claim that

$$\alpha_{lL} \ \sim_{\text{coe}} \ \lambda_l \boxtimes \tilde{\alpha}_{lL} \ \sim_{\text{conj}} \ \lambda_l \boxtimes \alpha_L. \tag{4}$$

$\boxtimes$ stands for the product action. Let us prove (4). Define $\varphi : X = \bigsqcup_{k=0}^{l-1} k + \tilde{X} \to \mathbb{Z}/l \times \tilde{X}$, $k + x \mapsto ([k], x)$. It is easy to see that the inverse of φ is given by $\psi : \mathbb{Z}/l \times \tilde{X} \to X$, $([k], x) \mapsto k + x$ for $0 \leq k \leq l-1$. Moreover, define $a : \mathbb{Z} \times X = \bigsqcup_{j=0}^{l-1} (j + l\mathbb{Z}) \times \bigsqcup_{k=0}^{l-1} (k + \tilde{X}) \to \mathbb{Z}/l \times l\mathbb{Z}$ by setting $a(j+h, k+x) = ([j], h)$ if $j + k \leq l-1$ and $a(j+h, k+x) = ([j], k+l)$ if $l < j+k$. Also, define $b : (\mathbb{Z}/l \times l\mathbb{Z}) \times (\mathbb{Z}/l \times \tilde{X}) \to \mathbb{Z}$ by setting $b(([j], h), ([k], x)) = j + h$ if $j + k \leq l-1$ and $b(([j], h), ([k], x)) = j + h - l$ if $l \leq j + k$, where $0 \leq j, k \leq l-1$. Then it is easy to check that φ, a, ψ and b satisfy (1) and (2), so that $\alpha_{lL} \ \sim_{\text{coe}} \ \lambda_l \boxtimes \tilde{\alpha}_{lL}$. Furthermore, $\lambda_l \boxtimes \tilde{\alpha}_{lL} \ \sim_{\text{conj}} \ \lambda_l \boxtimes \alpha_L$ is easy to see. This proves (4).
Now we can complete the proof for (iv) $\Rightarrow$ (i). Without loss of generality we may assume that $\sigma = \text{id}$, i.e., $m_i M_i = n_i N_i$ for all $1 \leq i \leq r$. Without loss of generality, we may further assume $\gcd(m_i, n_i) = 1$. Then we can write $M_i = n_i L_i$ and $N_i = m_i L_i$ for some supernatural number L_i. Set $L = \prod_{i=1}^r L_i$, and choose natural numbers m and n with $\gcd(m, L) = 1 = \gcd(n, L)$ such that $\prod_{i=1}^r M_i = (\prod_{i=1}^r n_i)(\prod_{i=1}^r L_i) = nL$ and $\prod_{j=1}^r N_j = (\prod_{j=1}^r m_j)(\prod_{j=1}^r L_j) = mL$. $\prod_{i=1}^r M_i = \prod_{j=1}^r N_j$ implies that $m = n$. Therefore, we get

$$\begin{aligned} \boxtimes_{i=1}^r \alpha_{M_i} &= \boxtimes_{i=1}^r \alpha_{n_i L_i} \overset{(4)}{\sim}_{\text{coe}} \boxtimes_{i=1}^r (\lambda_{n_i} \boxtimes \alpha_{L_i}) \overset{(4)}{\sim}_{\text{coe}} \alpha_{nL_1} \boxtimes \alpha_{L_2} \boxtimes \dots \boxtimes \alpha_{L_r} \\ &= \alpha_{mL_1} \boxtimes \alpha_{L_2} \boxtimes \dots \boxtimes \alpha_{L_r} \overset{(4)}{\sim}_{\text{coe}} \boxtimes_{j=1}^r \alpha_{N_j}. \end{aligned}$$

□

In contrast, for conjugacy, we get

Theorem 3.4. *Let I and J be finite sets. For supernatural numbers $\{M_i\}_{i \in I}$ and $\{N_j\}_{j \in J}$, $\mathbb{Z}^I \curvearrowright \prod_{i \in I} \mathbb{Z}/M_i \ \sim_{conj} \ \mathbb{Z}^J \curvearrowright \prod_{j \in J} \mathbb{Z}/N_j$ if and only if there exists a finite set K, supernatural numbers $\{L_k\}_{k \in K}$ such that*

- $I = \bigsqcup_{k \in K} I_k$, $J = \bigsqcup_{k \in K} J_k$,
- $|I_k| = |J_k|$ *for all* $k \in K$,
- *for every* $k \in K$, *every* $i \in I_k$ *and* $j \in J_k$, *we can write* $M_i = m_i L_k$ *and* $N_j = n_j L_k$ *for some (uniquely determined)* $m_i, n_j \in \mathbb{N}$ *with* $\gcd(m_i, L_k) = 1 = \gcd(n_j, L_k)$, *and we have* $\prod_{i \in I_k} \mathbb{Z}/m_i \cong \prod_{j \in J_k} \mathbb{Z}/n_j$.

Proof. "$\Rightarrow$": Assume that $\rho : \mathbb{Z}^I \cong \mathbb{Z}^J$ is a group isomorphism and $\varphi : \prod_{i \in I} \mathbb{Z}/M_i \cong \prod_{j \in J} \mathbb{Z}/N_j$ such that $\varphi(g.x) = \rho(g).\varphi(x)$. $\mathbb{Z}^I \cong \mathbb{Z}^J$ implies that $|I| = |J|$. Set $r = |I| = |J|$. Moreover, we may assume $\varphi(0) = 0$ (otherwise go over to $\varphi - \varphi(0)$). Let ρ be multiplication with $S \in GL_r(\mathbb{Z})$. It is straightforward to check that if $S_{j,i} \neq 0$, then $N_j \lesssim M_i$. So there exist a finite set K and decompositions $I = \bigsqcup_{k \in K} I_k$, $J = \bigsqcup_{k \in K} J_k$ with $|I_k| = |J_k|$ for all $k \in K$ such that for every $(i,j) \in I_k \times J_k$, $M_i \sim N_j$.
Fix $k \in K$. Find a supernatural number L_k such that for every $i \in I_k$, $j \in J_k$, there are $m_i \in \mathbb{N}$, $n_j \in \mathbb{N}$ with $\gcd(m_i, L_k) = 1 = \gcd(n_j, L_k)$ such that $M_i = m_i L_k$ and $N_j = n_j L_k$. Then φ restricts to an isomorphism of topological abelian groups

$$\varphi_k : (\prod_{i \in I_k} \mathbb{Z}/m_i) \times (\prod_{i \in I_k} \mathbb{Z}/L_k) \cong \prod_{i \in I_k} \mathbb{Z}/M_i \xrightarrow{\cong} \prod_{j \in J_k} \mathbb{Z}/N_j \cong (\prod_{j \in J_k} \mathbb{Z}/n_j) \times (\prod_{j \in J_k} \mathbb{Z}/L_k).$$

Let $l \in \mathbb{N}$ satisfy $\prod_{j \in J_k} n_j \mid l$ and $\gcd(l, L) = 1$. Certainly, $\varphi_k(w, 0)$ is of the form $(z, 0)$ as for all $i \in I_k$, there exists no non-zero homomorphism $\mathbb{Z}/m_i \to \mathbb{Z}/L_k$. Also, $\varphi_k(0, y)$ is of the form $\varphi_k(0, l \cdot \tilde{y}) = l \cdot \varphi_k(0, \tilde{y})$, hence of the form $(0, x)$ as for all $j \in J_k$, $l \equiv 0$ in $\mathbb{Z}/n_j$. Hence $\varphi_k = \phi_t \times \phi_L$ for some group isomorphisms $\phi_t : \prod_{i \in I_k} \mathbb{Z}/m_i \xrightarrow{\cong} \prod_{j \in J_k} \mathbb{Z}/n_j$ and $\phi_L : \prod_{i \in I_k} \mathbb{Z}/L_k \xrightarrow{\cong} \prod_{j \in J_k} \mathbb{Z}/L_k$.
"$\Leftarrow$": Without loss of generality we may assume $|K| = 1$. Let $K = \{k\}$, $I = I_k$, $J = J_k$, $|I| = |J| = r$. We may assume that $I = J = \{1, \dots, r\}$. Let $L = L_k$ be a supernatural number such that for every $1 \leq i, j \leq r$, $M_i = m_i L$ and $N_j = n_j L$ for some (unique) $m_i, n_j \in \mathbb{N}$ with $\gcd(m_i, L) = 1 = \gcd(n_j, L)$, and such that $\prod_{i=1}^r \mathbb{Z}/m_i \cong \prod_{j=1}^r \mathbb{Z}/n_j$. By the theory of elementary divisors, there are $S, T \in GL_r(\mathbb{Z})$ such that $S \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_r \end{pmatrix} T = \begin{pmatrix} n_1 & & 0 \\ & \ddots & \\ 0 & & n_r \end{pmatrix}$. Thus $S \left(\begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_r \end{pmatrix} \mathbb{Z}^r \right) = \begin{pmatrix} n_1 & & 0 \\ & \ddots & \\ 0 & & n_r \end{pmatrix} \mathbb{Z}^r$. So the same matrix S induces two group isomorphisms $\rho : \mathbb{Z}^r \to \mathbb{Z}^r$, $g \mapsto Sg$ and $\phi_t : \prod_{i=1}^r \mathbb{Z}/m_i \cong \mathbb{Z}^r \Big/ \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_r \end{pmatrix} \mathbb{Z}^r \cong \mathbb{Z}^r \Big/ \begin{pmatrix} n_1 & & 0 \\ & \ddots & \\ 0 & & n_r \end{pmatrix} \mathbb{Z}^r \cong \prod_{i=j}^r \mathbb{Z}/n_j$ as well as an isomorphism of topological groups $\phi_L : (\mathbb{Z}/L)^r \cong (\mathbb{Z}/L)^r$. Let φ be the isomorphism

$$\begin{aligned} \prod_{i=1}^r \mathbb{Z}/M_i &\cong \prod_{i=1}^r (\mathbb{Z}/m_i \times \mathbb{Z}/L) \cong (\prod_{i=1}^r \mathbb{Z}/m_i) \times (\mathbb{Z}/L)^r \\ \xrightarrow{\phi_t \times \phi_L} &\ (\prod_{j=1}^r \mathbb{Z}/n_j) \times (\mathbb{Z}/L)^r \cong \prod_{j=1}^r (\mathbb{Z}/n_j \times \mathbb{Z}/L) \cong \prod_{j=1}^r \mathbb{Z}/N_j. \end{aligned}$$

Then $\varphi|_{\mathbb{Z}^r} = \rho$, and so $\varphi(g.x) = \rho(g).\varphi(x)$ for all $g \in \mathbb{Z}^r$ and $x \in \prod_{i=1}^r \mathbb{Z}/M_i$. This means that $\mathbb{Z}^I \curvearrowright \prod_{i \in I} \mathbb{Z}/M_i \ \sim_{\text{conj}} \ \mathbb{Z}^J \curvearrowright \prod_{j \in J} \mathbb{Z}/N_j$. $\square$

Comparing Theorem 3.3 and Theorem 3.4, we can easily construct products of odometers which are continuously orbit equivalent but not conjugate.

Example 3.5. Let $r \geq 2$. Let p and q be primes, $p \neq q$, and let $n \in \mathbb{N}$ with $n > 1$ and $\gcd(p,n) = 1 = \gcd(q,n)$. If we set $M_1 = n \cdot p^\infty$, $M_2 = q^\infty$, $M_3 = \ldots = M_r = p^\infty$ and $N_1 = p^\infty$, $N_2 = n \cdot q^\infty$, $N_3 = \ldots = N_r = p^\infty$, then $\mathbb{Z}^r \curvearrowright \prod_{i=1}^r \mathbb{Z}/M_i \ \sim_{\mathrm{coe}}\ \mathbb{Z}^r \curvearrowright \prod_{j=1}^r \mathbb{Z}/N_j$ but $\mathbb{Z}^r \curvearrowright \prod_{i=1}^r \mathbb{Z}/M_i \not\sim_{\mathrm{conj}} \mathbb{Z}^r \curvearrowright \prod_{j=1}^r \mathbb{Z}/N_j$.

3.2. **Actions of non-abelian free groups.** Let us construct actions of the free group $\mathbb{F}_r$ ($r \geq 2$) on the Cantor set, which are continuously orbit equivalent but not conjugate. Let $a_1, \ldots, a_r$ be generators of $\mathbb{F}_r$. Let $\beta : \mathbb{F}_r \curvearrowright \partial\mathbb{F}_r$ be the $\mathbb{F}_r$-action on the Gromov boundary of $\mathbb{F}_r$, and set $\beta_i := \beta_{a_i}$. For a supernatural number M, let $\alpha_M : \mathbb{Z} \curvearrowright \mathbb{Z}/M$ be the corresponding odometer transformation. For supernatural numbers M_1, M_2, N_1 and N_2, define actions γ : $\mathbb{F}_r \curvearrowright \partial\mathbb{F}_r \times (\mathbb{Z}/M_1) \times (\mathbb{Z}/M_2)$ and $\delta : \mathbb{F}_r \curvearrowright \partial\mathbb{F}_r \times (\mathbb{Z}/N_1) \times (\mathbb{Z}/N_2)$ by setting $\gamma_1 := \beta_1 \times \alpha_{M_1} \times \mathrm{id}$, $\gamma_2 := \beta_2 \times \mathrm{id} \times \alpha_{M_2}$, $\gamma_i := \beta_i \times \mathrm{id} \times \mathrm{id}$ for all $i \geq 3$, $\gamma_{a_i} = \gamma_i$ for all $1 \leq i \leq r$, and similarly $\delta_1 := \beta_1 \times \alpha_{N_1} \times \mathrm{id}$, $\delta_2 := \beta_2 \times \mathrm{id} \times \alpha_{N_2}$, $\delta_i := \beta_i \times \mathrm{id} \times \mathrm{id}$ for all $i \geq 3$, $\delta_{a_i} = \delta_i$ for all $1 \leq i \leq r$.

Theorem 3.6. *Let p and q be primes, $p \neq q$ and $n \in \mathbb{N}$ with $n > 1$, $\gcd(p,n) = 1 = \gcd(q,n)$. If we set $M_1 = n \cdot p^\infty$, $M_2 = q^\infty$ and $N_1 = p^\infty$, $N_2 = n \cdot q^\infty$, then $\gamma \ \sim_{coe}\ \delta$ but $\gamma \not\sim_{conj} \delta$.*

For the proof, we need some preparation. Let X be a totally disconnected compact space. Later on, X will be the Cantor space. Let $C^\infty(X, \mathbb{C}) = C(X, \mathbb{Z}) \otimes \mathbb{C}$. We have an isomorphism $C^\infty(X, \mathbb{C}) \ \overset{\cong}{\longrightarrow}\ \left\{X \xrightarrow{f} \mathbb{C} \text{ continuous: } f(X) \subseteq \mathbb{C} \text{ finite}\right\}$, $f \otimes z \ \mapsto\ f \cdot z$. Here $(f \cdot z)(x) = f(x)z$. Let us view elements in $C^\infty(X, \mathbb{C})$ as $\mathbb{C}$-valued continuous functions on X via this explicit isomorphism. Let $\phi :\ X \to X$ be a homeomorphism, and denote the induced automorphism of $C(X)$ by ϕ again. Obviously, $\phi(C^\infty(X, \mathbb{C})) \subseteq C^\infty(X, \mathbb{C})$. We define $E(\phi) := \{z \in \mathbb{T}\colon \phi(f) = zf \text{ for some } 0 \neq f \in C^\infty(X, \mathbb{C})\}$. Let $g_1, \ldots, g_r$ be generators of $\mathbb{F}_r$, $g = g_1$, Y a totally disconnected compact space and $\alpha :\ Y \to Y$ a homeomorphism.

Proposition 3.7. *For $\phi = \beta_g \times \alpha$, we have $E(\phi) = E(\alpha)$.*

Proof. We think of elements in $\partial\mathbb{F}_r$ as infinite reduced words in $g^{\pm 1} = g_1^{\pm 1}, g_2^{\pm 1}, \ldots, g_r^{\pm 1}$. Let W be the set of finite reduced words in $g_1^{\pm 1}, g_2^{\pm 1}, \ldots, g_r^{\pm 1}$ which do not end on g_r^{-1} nor on g_r^2. For $w \in W$, let C_w be the subspace of $\partial\mathbb{F}_r$ consisting of those infinite reduced words which start with w. Note that the empty word $\emptyset$ lies in W, and that $C_\emptyset = \partial\mathbb{F}_r$. Clearly, $\{1_{C_w}\colon w \in W\}$ is a $\mathbb{Z}$-basis for $C(\partial\mathbb{F}_r, \mathbb{Z})$. Take a family $\mathcal{C}$ of compact open subsets of Y such that $\{1_C\colon C \in \mathcal{C}\}$ is a $\mathbb{Z}$-basis for $C(Y, \mathbb{Z})$. Then $\{1_{C_w} \otimes 1_C\colon w \in W, C \in \mathcal{C}\}$ is a $\mathbb{Z}$-basis for $C(\partial\mathbb{F}_r \times Y, \mathbb{Z}) \cong C(\partial\mathbb{F}_r, \mathbb{Z}) \otimes C(Y, \mathbb{Z})$.

Let $z \in E(\phi)$, and let $f = \sum_{w \in W_1} 1_{C_w} \otimes 1_{C(w)} \otimes \lambda(w)$ ($\lambda(w) \in \mathbb{C} \setminus \{0\}$ for all $w \in W_1$) be a nonzero element in $C(\partial\mathbb{F}_r \times Y, \mathbb{C}) \cong C(\partial\mathbb{F}_r, \mathbb{Z}) \otimes C(Y, \mathbb{Z}) \otimes \mathbb{C}$ with $\phi(f) = zf$. Here W_1 is a finite subset of W. Then $\sum_{w \in W_1} 1_{C_w} \otimes (1_{C(w)} \otimes z\lambda(w)) = zf = \phi(f) = \sum_{w \in W_1} 1_{\beta_g(C_w)} \otimes (1_{\alpha(C(w))} \otimes \lambda(w)) = \sum_{w \in W_2} 1_{C_w} \otimes f(w)$ for some $0 \neq f(w) \in C(Y, \mathbb{Z}) \otimes \mathbb{C}$ for $w \in W_2$. Here $W_2 \subseteq W$ is a finite subset such that $1_{\beta_g(C_w)} \in \mathbb{Z}\text{-span}(W_2)$ for all $w \in W_1$. It follows that $W_1 = W_2 \supseteq \{gw\colon \emptyset \neq w \neq g^{-1}, w \in W_1\} \cup \{g\}$ if $g^{-1} \in W_1$ and $W_1 = W_2 \supseteq \{gw\colon \emptyset \neq w \neq g^{-1}, w \in W_1\}$ if $g^{-1} \notin W_1$. Here we use that $\beta_g(C_w) = C_{gw}$ if $\emptyset \neq w \neq g^{-1}$ and $\beta_g(C_{g^{-1}}) = \partial\mathbb{F}_r \setminus C_g$.

We claim that it already follows that $W_1 = \{\partial\mathbb{F}_r\}$: If there is $w \in W_1$ not starting with g^{-1}, then $g^n w \in W_1$ for all $m \in \mathbb{N}$ which is impossible since W_1 is finite. If there is $w \in W_1$

of the form $g^{-m}v$ where $v \neq \emptyset$ is a finite reduced word not starting with $g^{\pm 1}$, then $v \in W_1$ contradicting our first observation. If there is $g^{-m} \in W_1$ for some $m \geq 1$, then $g^{-1} \in W_1$, hence $g \in W_1$. This again contradicts our first observation. Therefore, the only possibility is $W_1 = \{\partial \mathbb{F}_r\}$.
Hence $f = 1 \otimes \tilde{f}$ for some $\tilde{f} \in C(Y, \mathbb{Z}) \otimes \mathbb{C}$. So $1 \otimes z\tilde{f} = zf = \phi(f) = 1 \otimes \alpha(\tilde{f})$. Hence it follows that $z\tilde{f} = \alpha(\tilde{f})$. This shows that $z \in E(\alpha)$. Since $z \in E(\phi)$ was arbitrary, we obtain $E(\phi) \subseteq E(\alpha)$. The reverse inclusion is obvious. □

We are now ready for the

Proof of Theorem 3.6. By [30, Theorem 5.10 and Proposition 5.2], $\gamma \sim_{\text{coe}} \delta$. So we just have to show $\gamma \not\sim_{\text{conj}} \delta$. Assume that there exists a homeomorphism $\varphi : \partial \mathbb{F}_r \times (\mathbb{Z}/M_1) \times (\mathbb{Z}/M_2) \xrightarrow{\cong} \partial \mathbb{F}_r \times (\mathbb{Z}/N_1) \times (\mathbb{Z}/N_2)$ and a group isomorphism $\rho : \mathbb{F}_r \cong \mathbb{F}_r$ such that $\varphi \circ \gamma_a = \delta_{\rho(a)} \circ \varphi$ for all $a \in \mathbb{F}_r$. Let $a := a_1$ and $g := \rho(a_1)$. Then in particular, $\varphi \circ \gamma_a = \delta_g \circ \varphi$, so that γ_a and δ_g are conjugate, and hence $E(\gamma_a) = E(\delta_g)$.
By construction, there are $k, l \in \mathbb{Z}$ with $\delta_g = \beta_g \times \alpha_{N_1}^k \times \alpha_{N_2}^l$, where $N_1 = p^\infty$ and $N_2 = n \cdot q^\infty$. Proposition 3.7 yields $E(\delta_g) = E(\alpha_{N_1}^k \times \alpha_{N_2}^l)$. For a supernatural number M, let $\mathbb{T}(M) = \mathbb{Z}[M^{-1}]/\mathbb{Z} \subseteq \mathbb{Q}/\mathbb{Z} \subseteq \mathbb{R}/\mathbb{Z} \cong \mathbb{T}$. If $l = 0$, then $E(\delta_g) = E(\alpha_{N_1}^k \times \mathrm{id}) = E(\alpha_{N_1}^k) = E(\alpha_{p^\infty}^k)$ is equal to $\{1\}$ or $\mathbb{T}(p^\infty)$. If $l \neq 0$, then $\mathbb{T}(q^\infty) = E(\alpha_{q^\infty}^l) \subseteq E(\alpha_{n \cdot q^\infty}^l) \subseteq E(\alpha_{p^\infty}^k \times \alpha_{n \cdot q^\infty}^l) = E(\alpha_{N_1}^k \times \alpha_{N_2}^l) = E(\delta_g)$. However, $E(\gamma_a) = E(\alpha_{M_1} \times \mathrm{id}) = E(\alpha_{n \cdot p^\infty}) = \mathbb{T}(n \cdot p^\infty)$ is not equal to $\{1\}$ nor $\mathbb{T}(p^\infty)$ and does not contain $\mathbb{T}(q^\infty)$. Hence $E(\gamma_a) \neq E(\delta_g)$. This is a contradiction. □

4. From continuous cocycle rigidity to continuous orbit equivalence rigidity

We introduce the notion of continuous cocycle rigidity. Let $G \curvearrowright X$ be a topological dynamical system and let H be a group.

Definition 4.1. *A continuous H-cocycle for $G \curvearrowright X$ is a continuous map $a : G \times X \to H$ such that $a(g_1 g_2, x) = a(g_1, g_2.x)a(g_2, x)$ for all $g_1, g_2 \in G$, $x \in X$.*

In other words, $a : G \times X \to H$ is a groupoid homomorphism, where we view $G \times X$ as a groupoid by identifying it with the transformation groupoid $G \ltimes X$ attached to $G \curvearrowright X$, and view H as the groupoid whose unit space is a point.

Definition 4.2. *Continuous H-cocycles a and a' for $G \curvearrowright X$ are continuously cohomologous ($a \sim a'$) if there exists a continuous map $u : X \to H$ such that $a(g, x) = u(g.x)a'(g, x)u(x)^{-1}$ for all $g \in G$ and $x \in X$.*

Definition 4.3. *$G \curvearrowright X$ is continuous H-cocycle rigid if for every continuous H-cocycle a for $G \curvearrowright X$, there exists a group homomorphism $\rho : G \to H$ such that $a \sim \rho$.*

Here we view ρ as the cocycle $G \times X \to H$, $(g, x) \mapsto \rho(g)$.

The following observation provides a first link between continuous cocycle rigidity and continuous orbit equivalence rigidity.

Proposition 4.4. *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free systems. Assume that $G \curvearrowright X \sim_{coe} H \curvearrowright Y$, and let φ, ψ, a and b be as in Definition 2.5. If there exists a continuous map $u : X \to H$ and a group isomorphism $\rho : G \to H$ such that $a(g, x) = u(g.x)\rho(g)u(x)^{-1}$ for* *all $g \in G$, $x \in X$, then ${}_u\varphi : X \to Y$, $x \mapsto u(x)^{-1}.\varphi(x)$ and ρ give rise to a conjugacy between $G \curvearrowright X$ and $H \curvearrowright Y$.*

Proof. ${}_u\varphi$ is obviously continuous, and an easy computation shows that ${}_u\varphi(g.x) = \rho(g).{}_u\varphi(x)$ for all $g \in G$, $x \in X$. It remains to show that ${}_u\varphi$ is a homeomorphism.
Let $\sigma = \rho^{-1}$, define $v : Y \to G$, $y \mapsto (\sigma(u(\psi(y))))^{-1}$ and $\tilde{b} : H \times Y \to G$, $(h, y) \mapsto v(h.y)\sigma(h)v(y)^{-1}$. Since

$$\begin{aligned} & \tilde{b}(a(g,x),\varphi(x)) = v(a(g,x).\varphi(x))\sigma(a(g,x))v(\varphi(x))^{-1} = v(\varphi(g.x))\sigma(a(g,x))v(\varphi(x))^{-1} \\ = \; & \sigma(u(g.x))^{-1}\sigma(a(g,x))\sigma(u(x)) = \sigma(u(g.x))^{-1}a(g,x)u(x)) = \sigma(\rho(g)) = g, \end{aligned}$$

Lemma 2.10 implies that $b = \tilde{b}$. Set ${}_v\psi : Y \to X$, $y \mapsto v(y)^{-1}\psi(y)$. ${}_v\psi$ is obviously continuous, and an easy computation shows that ${}_v\psi(h.y) = \rho(h).{}_v\psi(y)$ for all $h \in H$, $y \in Y$. Moreover,

$$\begin{aligned} {}_v\psi({}_u\varphi(x)) &= {}_v\psi(u(x)^{-1}.\varphi(x)) = \sigma(u(x)^{-1}).{}_v\psi(\varphi(x)) \\ &= \sigma(u(x))^{-1}v(\varphi(x))^{-1}.x = \sigma(u(x))^{-1}\sigma(u(x)).x = x, \\ \textit{and } {}_u\varphi({}_v\psi(y)) &= {}_u\varphi(v(y)^{-1}.\psi(y)) = \rho(v(y)^{-1})_u\varphi(\psi(y)) \\ &= \rho(v(y))^{-1}u((\psi(y))^{-1}.y = u(\psi(y))u((\psi(y))^{-1}.y = y. \end{aligned}$$

Thus ${}_u\varphi$ is a homeomorphism, with inverse ${}_v\psi$, and the proof is complete. □

Continuous cocycle rigidity means that every cocycle, whether or not if comes from a continuous orbit equivalence, is continuously cohomologous to a group homomorphism. At the same time, the preceding proposition shows that for continuous orbit equivalence rigidity, cocycles are required to be continuously cohomologous to group isomorphisms. Therefore, there does not seem to be any obvious connection between continuous cocycle rigidity and continuous orbit equivalence rigidity. However, we have the following

Theorem 4.5. *Suppose that G is amenable and torsion-free. Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free systems on compact spaces X and Y. Assume that $G \curvearrowright X \sim_{coe} H \curvearrowright Y$, and let $a : G \times X \to H$ be as in Definition 2.5. If $a \sim \rho$ for some group homomorphism $\rho : G \to H$, then ρ must be an isomorphism.*

Proof. Let $u : X \to H$ be continuous such that $a(g,x) = u(g.x)\rho(g)u(x)^{-1}$ for all $g \in G$, $x \in X$. Take $x \in X$ with $G_x = \{e\}$. Then by Lemma 2.11, $a_x : G \to H$, $g \mapsto a(g,x)$ is bijective. Let $u_x : G \to H$, $g \mapsto u(g.x)$. Then $a_x(g) = u_x(g)\rho(g)u_x(e)^{-1}$. u is continuous and X is compact, hence $u(X) \subseteq H$ is finite. In particular, $u_x(G)$ is finite. Therefore, for every $g \in \ker(\rho)$, $a_x(g) \in u_x(G)u_x(e)^{-1}$. It follows that $a_x(\ker(\rho))$ is finite. Since a_x is injective, $\ker(\rho)$ is finite. But G is torsion-free. This implies $\ker(\rho) = \{e\}$, so that ρ is injective.
It remains to prove surjectivity. Since a_x is surjective, we have $H = u(X)\rho(G)u(x)^{-1} = u(X)\rho(G)$. Thus, $[H : \rho(G)] < \infty$. In particular, H is also amenable. Without loss of generality, we may assume $u_x(e) = u(x) = e$. Otherwise, replace ρ by $u(x)\rho u(x)^{-1}$ and u_x by $u_x \cdot u(x)^{-1}$. Suppose that $\rho(G) \subsetneq H$. Let R be a complete system of left coset representatives of $\rho(G)$ in H. Since H is amenable, there exists a finite subset F of H such that $|rF \triangle F| < \frac{1}{3}|F|$ for all $r \in R$ and $|sF \triangle F| < \frac{1}{3|u(X)|}|F|$ for all $s \in u(X)$.
Assume that $|F \cap \rho(G)| > \frac{2}{3}|F|$. By assumption ($\rho(G) \subsetneq H$), there exists $r \in R$ with $r\rho(G) \cap \rho(G) = \emptyset$. So $r(F \cap \rho(G)) \cap (F \cap \rho(G)) = \emptyset$, and we obtain $|r(F \cap \rho(G)) \cap F| < \frac{1}{3}|F|$. Moreover, $|rF \setminus r(F \cap \rho(G))| = |F \setminus (F \cap \rho(G))| < \frac{1}{3}|F|$. Therefore, $\frac{2}{3}|F| < |rF \cap F| \leq$

$|(rF \setminus r(F \cap \rho(G))) \cup (r(F \cap \rho(G)) \cap F)| < \frac{1}{3}|F| + \frac{1}{3}|F| = \frac{2}{3}|F|$. But this is a contradiction. Therefore, we must have $|F \cap \rho(G)| \leq \frac{2}{3}|F|$.

We certainly have $a_x^{-1}(F) \subseteq \rho^{-1}(\bigcup_{s \in u(X)} s^{-1}F)$. Hence

$$\begin{aligned}|F| &= \left|a_x^{-1}(F)\right| \leq \left|\rho^{-1}(\bigcup_{s \in u(X)} s^{-1}F)\right| \leq \left|\rho^{-1}(F \cup (\bigcup_{s \in u(X)} s^{-1}F) \setminus F)\right| \\ &\leq \left|\rho^{-1}(F)\right| + \sum_{s \in u(X)} \left|s^{-1}F \setminus F\right| < |F \cap \rho(G)| + \frac{1}{3}|F| \leq |F|.\end{aligned}$$

This is a contradiction. We conclude that $\rho(G) = H$. □

Clearly, Proposition 4.4 and Theorem 4.5 imply

Theorem (Theorem 1.7)**.** *Let G be a torsion-free amenable group. Assume that $G \curvearrowright X$ and $H \curvearrowright Y$ are topologically free systems on compact spaces X and Y, and suppose that $G \curvearrowright X$ and $H \curvearrowright Y$ are continuously orbit equivalent. If $G \curvearrowright X$ is continuous H-cocycle rigid, then $G \curvearrowright X$ and $H \curvearrowright Y$ must be conjugate.*

5. Continuous cocycle rigidity via group cohomology

The first goal of this section is to rephrase continuous cocycle rigidity in the language of non-abelian group cohomology. For the sake of completeness, we briefly recall the definition of non-abelian group cohomology (H^1). We refer the reader to [27, Part Three, Appendix "Non-abelian cohomology"] and [28, Chapter I, § 5] for details.

Let G be a group acting on a group A by automorphisms, denoted by $G \times A \to A$, $(s, a) \mapsto s.a$. A 1-cocycle of G in A is a map $G \to A$, $s \mapsto a_s$ such that $a_{st} = a_s s.a_t$. We write $Z^1(G, A)$ for the set of all these 1-cocycles. Given 1-cocycles a and a' of G in A, we say that a is cohomologous to a' ($a \sim a'$) if there exists $b \in A$ with $a'_s = b^{-1} a_s s.b$ for all $s \in G$. We define $H^1(G, A) := Z^1(G, A)/\sim$. Clearly, $H^1(G, A)$ is (covariantly) functorial in A.

Proposition 5.1. *Let $G \curvearrowright X$ be a topological dynamical system on a compact space X. Let H be a group.*

$G \curvearrowright X$ is continuous H-cocycle rigid if and only if the canonical map $H \to C(X, H)$ (the map dual to $X \to \{\mathrm{pt}\}$) induces a surjective map $H^1(G, H) \to H^1(G, C(X, H))$.

Note that we equip H with the trivial G-action, and G acts on $C(X, H)$ via $(s.a)(x) = a(s^{-1}.x)$.

Proof. Just check that $c: C(G, C(X, H)) \to C(G \times X, H)$ defined by $c(a)(g, x) = a_g(g, x)$ is a bijection, with inverse given by $c^{-1}(b)_s(x) = b(s, s^{-1}.x)$. c identifies $Z^1(G, C(X, H))$ with the set of continuous H-cocycles for $G \curvearrowright X$ in the sense of Definition 4.1. In addition, $a \sim a'$ if and only if $c(a) \sim c(a')$ in the sense of Definition 4.2. It is then easy to see that for $a \in Z^1(G, C(X, H))$, the class $[a] \in H^1(G, C(X, H))$ lies in the image of the canonical map $H^1(G, H) \to H^1(G, C(X, H))$ if and only if $c(a) \sim \rho$ for some group homomorphism $\rho: G \to H$. □

Using the language of non-abelian group cohomology, we now prove a positive result in continuous cocycle rigidity.

Proposition 5.2. *Let $G \curvearrowright X$ be a topological dynamical system on a compact space X, and suppose that $C(X,\mathbb{Z}) = \mathbb{Z} \cdot 1 \oplus N$ as $\mathbb{Z}G$-modules. Let H be an abelian G-group. Then the canonical map $H \to C(X,H)$ induces injections $H^i(G,H) \hookrightarrow H^i(G,C(X,H))$ for every $i \geq 0$. Moreover, assume that for every G-module M, $H^1(G,N \otimes M) \cong \{0\}$. Then the canonical map $H \to C(X,H)$ induces a surjection $H^1(G,H) \twoheadrightarrow H^1(G,C(X,H))$.*
In particular, $G \curvearrowright X$ is continuous H-cocycle rigid.

Proof. By assumption, $C(X,H) \cong C(X,\mathbb{Z}) \otimes H \cong H \oplus (N \otimes H)$. Therefore, for every $i \geq 0$, $C(G,H) \to C(G,C(X,H))$ induces injective maps $H^i(G,H) \hookrightarrow H^i(G,C(X,H))$ as these maps correspond to the canonical inclusions $H^i(G,H) \hookrightarrow H^i(G,H) \oplus H^i(G,N \otimes H)$ under the identification $H^i(G,C(X,H)) \cong H^i(G,H) \oplus H^i(G,N \otimes H)$. Moreover, it is clear that for $i = 1$, $H^1(G,H) \to H^1(G,C(X,H))$ is surjective if and only if $H^1(G,N \otimes H) \cong \{0\}$.
Applying this to an abelian group H with trivial G-action, we obtain that $G \curvearrowright X$ is continuous H-cocycle rigid by Proposition 5.1. □

Lemma 5.3. *Suppose that $1 \to H' \xrightarrow{\iota} H \xrightarrow{\pi} H'' \to 1$ is an exact sequence of groups, and assume that H' is abelian. Let $G \curvearrowright X$ be a topological dynamical system on a compact space X, and suppose that $C(X,\mathbb{Z}) \cong \mathbb{Z} \cdot 1 \oplus N$ as $\mathbb{Z}G$-modules. Moreover, assume that for every G-module M, $H^1(G,N \otimes M) \cong \{0\}$.*

If $G \curvearrowright X$ is continuous H''-cocycle rigid, then $G \curvearrowright X$ is continuous H-cocycle rigid.

Proof. Write $C' = C(X,H')$, $C = C(X,H)$ and $C'' = C(X,H'')$. Let $i : C' \to C$ and $p : C \to C''$ be the homomorphisms induced by ι and π. We get the following commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & H' \arrow[r, "\iota"] \arrow[d, "\varphi'"] & H \arrow[r, "\pi"] \arrow[d, "\varphi"] & H'' \arrow[r] \arrow[d, "\varphi''"] & 1 \\
1 \arrow[r] & C' \arrow[r, "i"] & C \arrow[r, "p"] & C'' \arrow[r] & 1
\end{tikzcd}$$

where φ', φ and φ'' are the canonical homomorphisms.
Take $x \in H^1(G,C)$. Since φ''_* is surjective, we can find $\xi'' \in H^1(G,H'')$ with $\varphi''_*(\xi'') = p_*(x)$. Let us first prove the following

Claim: There exists $\zeta \in H^1(G,H)$ with $\pi_*(\zeta) = \xi''$.

Proof of the claim: Let λ'' be a 1-cocycle of G in H'' representing ξ''. Lift λ'' to a map $\lambda : G \to H$ such that $\pi \circ \lambda = \lambda''$. Then, as in [28, Chapter I, § 5.6], define a 2-cocycle λ' of G in H' by setting $\lambda'_{s,t} := \lambda_s s.\lambda_t \lambda_{st}^{-1}$ and let $\Delta(\lambda'') := [\lambda'] \in H^2(G,H')$. Since $p \circ \varphi \circ \lambda = \varphi'' \circ \pi \circ \lambda = \varphi'' \circ \lambda''$, $\varphi \circ \lambda$ is a lift of $\varphi'' \circ \lambda''$. Moreover, $[\varphi'' \circ \lambda''] = \varphi''_*[\lambda''] = \varphi''_*(\xi'') = p_*(x)$ lies in $\mathrm{Im}\,(p_*)$. Therefore, by [28, Chapter I, § 5.6, Proposition 41], $\Delta(\varphi'' \circ \lambda'') = 0$ in $H^2(G,C')$. Hence $0 = \Delta(\varphi'' \circ \lambda'') = [\varphi' \circ \lambda'] = \varphi'_*[\lambda'] = \varphi'_*(\Delta(\lambda''))$. As φ'_* is injective by Proposition 5.2, we obtain $\Delta(\lambda') = 0$. And thus, again by [28, Chapter I, § 5.6, Proposition 41], ξ'' lies in $\mathrm{Im}\,(\pi_*)$. This proves our claim.
So we can find $\zeta \in H^1(G,H)$ with $\pi_*(\zeta) = \xi''$. Then $p_*(\varphi_*(\zeta)) = \varphi''_*(\pi_*(\zeta)) = \varphi''_*(\xi'') = p_*(x)$. Let β be a 1-cocycle of G in H representing ζ. β is a group homomorphism $G \to H$, $s \mapsto \beta_s$. We define a G-action on H by setting $s_\bullet h := \beta_s h \beta_s^{-1}$. Since H' is a normal subgroup of H, this G-action restricts to a G-action on H'. We write ${}_\beta H'$ for the G-group H' with respect to this new G-action. Let $b = \varphi(\beta)$. We obtain a new G-group structure on $C' = C(X,H')$ given by $s_\bullet c = b_s s.c b_s^{-1}$, and we write ${}_b C'$ for the new G-group obtained in this way. Now let a be

a 1-cocycle of G in C representing x. $p_*(\varphi_*(\zeta)) = p_*(x)$ implies that $p(a) \sim p(b)$, i.e., there is $c \in C$ such that $p(c^{-1} a_s s.c) = p(b_s)$ for all $s \in G$. Replacing a by $G \to C$, $s \mapsto c^{-1} a_s s.c$, we may assume that $p(a_s) = p(b_s)$ for all $s \in G$. Hence $a_s b_s^{-1} \in C'$ for all $s \in G$. Moreover,

$$a_{st} b_{st}^{-1} = a_s s.a_t b_t^{-1} b_s^{-1} = a_s b_s^{-1} b_s (s.a_t b_t^{-1}) b_s^{-1} = (a_s b_s^{-1}) s_{\bullet} (a_t b_t^{-1})$$

shows that $ab^{-1} : G \to {}_b C'$, $s \mapsto a_s b_s^{-1}$ is a 1-cocycle of G in ${}_b C'$. By Proposition 5.2 the canonical map $H^1(G, {}_\beta H') \to H^1(G, {}_b C')$ is surjective, so that there is a 1-cocycle γ of G in ${}_\beta H'$, $\gamma : G \to {}_\beta H'$, $s \mapsto \gamma_s$ and $c' \in C'$ such that $(c')^{-1} a_s b_s^{-1} s_{\bullet} c' = \varphi'(\gamma_s)$ for all $s \in G$. It follows that $(c')^{-1} a_s b_s^{-1} b_s s.c' b_s^{-1} = \varphi'(\gamma_s)$ and thus

$$(c')^{-1} a_s s.c' = \varphi'(\gamma_s) b_s = \varphi(\gamma_s \beta_s). \tag{5}$$

Note that $\gamma\beta : G \to H$, $s \mapsto \gamma_s \beta_s$ is a 1-cocycle of G in H, i.e., a group homomorphism, since

$$\gamma_{st} \beta_{st} = \gamma_s s_{\bullet} \gamma_t \beta_s \beta_t = \gamma_s \beta_s \gamma_t \beta_s^{-1} \beta_s \beta_t = (\gamma_s \beta_s)(\gamma_t \beta_t).$$

By (5), we know that $a \sim \varphi(\gamma\beta)$. This shows that $x \in \mathrm{Im}\,(\varphi_*)$. As x was arbitrary, we conclude that φ_* is surjective. $\square$

Remark 5.4. The latter part of the proof of Lemma 5.3 (the part after the proof of the claim) may be phrased using the notion of twistings, as explained in [28, Chapter I, § 5.3] and at the beginning of [28, Chapter I, § 5.4]. However, the present proof has the advantage of being more elementary. I am indebted to the referee who pointed this out.

Corollary 5.5. *Let $G \curvearrowright X$ be a topological dynamical system on a compact space X. Suppose that $C(X, \mathbb{Z}) \cong \mathbb{Z} \cdot 1 \oplus N$ as $\mathbb{Z}G$-modules. Moreover, assume that for every G-module M, $H^1(G, N \otimes M) \cong \{0\}$.*

Then $G \curvearrowright X$ is continuous H-cocycle rigid for every solvable group H.

Proof. We proceed inductively on the length of a series $\{1\} = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_n = H$ with $H_i \triangleleft H$ for all $1 \leq i \leq n$ and H_i / H_{i-1} abelian for all $1 \leq i \leq n$. The case $n = 1$ is taken care of by Proposition 5.2. To go from $n-1$ to n, consider the series $\{1\} = H_1/H_1 \subseteq H_2/H_1 \subseteq \ldots \subseteq H_n/H_1 = H/H_1$. By induction hypothesis, $G \curvearrowright X$ is continuous H/H_1-cocycle rigid. Applying Lemma 5.3 to $1 \to H_1 \to H \to H/H_1 \to 1$, we obtain that $G \curvearrowright X$ is continuous H-cocycle rigid. $\square$

Now we would like to find examples of topological systems $G \curvearrowright X$ on compact spaces X such that $C(X, \mathbb{Z}) = \mathbb{Z} \cdot 1 \oplus N$ as $\mathbb{Z}G$-modules and $H^1(G, N \otimes M) \cong \{0\}$ for every G-module M. Here is a general criterion:

Lemma 5.6. *Let G be a duality group in the sense of* [3, Chapter VIII, § 10]*, and let $G \curvearrowright X$ be a topological system on a compact space X such that*

$$C(X, \mathbb{Z}) = \mathbb{Z} \cdot 1 \oplus N \text{ as } \mathbb{Z}G\text{-modules, with } \mathrm{pd}_{\mathbb{Z}G}(N) < \mathrm{cd}(G) - 1. \tag{6}$$

Then $H^1(G, N \otimes M) \cong \{0\}$ for every G-module M.

Proof. Set $n := \mathrm{cd}(G)$. Then, as G is a duality group, we have $H^1(G, N \otimes M) \cong H_{n-1}(G, N \otimes D \otimes M)$, where D is the dualizing module of G as in [3, Chapter VIII, § 10]. But because $\mathrm{pd}_{\mathbb{Z}G}(N) < n - 1$, we must have $H_{n-1}(G, N \otimes D \otimes M) \cong \{0\}$, and our claim follows. $\square$

Clearly, Corollary 5.5 and Lemma 5.6 imply

Theorem (Theorem 1.8). *Let G be a duality group, and let $G \curvearrowright X$ be a topological system on a compact space X such that $C(X,\mathbb{Z}) = \mathbb{Z}\cdot 1 \oplus N$ as $\mathbb{Z}G$-modules, with $\mathrm{pd}_{\mathbb{Z}G}(N) < \mathrm{cd}(G) - 1$. Then $G \curvearrowright X$ is continuous H-cocycle rigid for every solvable group H.*

Let us now present some concrete examples satisfying (6).

Definition 5.7. *A topological dynamical system $G \curvearrowright X$ on a compact space X is called $\mathbb{Z}G$-free if $C(X,\mathbb{Z}) = \mathbb{Z}\cdot 1 \oplus N$ as $\mathbb{Z}G$-modules, where N is a free $\mathbb{Z}G$-module.*

Remark 5.8. It is easy to see that $G \curvearrowright X$ is $\mathbb{Z}G$-free if we can find a $\mathbb{Z}$-basis $\mathcal{B}$ for $C(X,\mathbb{Z})$ with the following properties:

- $\mathcal{B}$ is G-invariant,
- $1_X \in \mathcal{B}$,
- G acts freely on $\mathcal{B} \setminus \{1_X\}$.

Clearly, a $\mathbb{Z}G$-free system satisfies (6) if $\mathrm{cd}(G) > 1$. Here are two classes of $\mathbb{Z}G$-free systems:

Example 5.9. Let G be a torsion-free group and X_0 a compact space. Then the Bernoulli action $G \curvearrowright X_0^G$ is $\mathbb{Z}G$-free. Namely, choose a $\mathbb{Z}$-basis $\mathcal{B}_0$ for $C(X_0,\mathbb{Z})$ with $1_{X_0} \in \mathcal{B}_0$. This is always possible, see for instance [6, Proposition 2.12]. Then set

$$\mathcal{B} = \left\{ \left(\bigotimes_{f\in F} b_f \right) \otimes 1_{X_0^{G\setminus F}} \colon F \subseteq G \text{ finite}, b_f \in \mathcal{B}_0 \right\}.$$

$\mathcal{B}$ is a $\mathbb{Z}$-basis as $C(X_0^G,\mathbb{Z}) = \bigcup_{F\subseteq G \text{ finite}} \left(\bigotimes_{f\in F} C(X_0) \right) \otimes 1_{X_0^{G\setminus F}}$. Obviously, $\left\{ 1_{X_0^G} \right\}$ lies in $\mathcal{B}$. Moreover, G acts freely on $\mathcal{B} \setminus \left\{ 1_{X_0^G} \right\}$ as G is torsion-free.

Building on the previous example, we now show that for torsion-free groups, subshifts of full shifts over finite alphabets whose forbidden words avoid a fixed letter are $\mathbb{Z}G$-free.

Example 5.10. Let G be a torsion-free group, $A = \{0,\dots,N\}$ a finite alphabet and $G \curvearrowright A^G$ the full shift. Elements in A^G are of the form $x = (x_\gamma)_{\gamma\in G}$, and $g \in G$ acts by $(g.x)_\gamma = x_{g^{-1}\gamma}$. For every G-invariant closed subset X of A^G we can find a collection $\{F_i\}_{i\in I}$ of non-empty finite subsets of G and $x_i \in A^{F_i}$, $i \in I$, such that

$$X = \left\{ x = (x_\gamma)_\gamma \in A^G \colon \text{For every } i \in I \text{ and } g \in G, \pi_{F_i}(g.x) \neq x_i \right\}.$$

Here π_{F_i} is the canonical projection $A^G \twoheadrightarrow A^{F_i}$. $\{x_i\}_{i\in I}$ are called the forbidden words for X.

Now assume that X is a G-invariant closed subset whose forbidden words x_i satisfy $x_i \in \{1,\dots,N\}^{F_i}$ for some finite subsets $F_i \subseteq G$, i.e., all the forbidden words avoid a fixed letter (0 in our case). If that is the case, then we claim that $G \curvearrowright X$ is $\mathbb{Z}G$-free.

Here is the reason: Obviously, $\mathcal{B}_0 = \left\{ 1_A, 1_{\{1\}}, \dots, 1_{\{N\}} \right\}$ is a $\mathbb{Z}$-basis for $C(A,\mathbb{Z})$. Given a finite subset $\emptyset \neq F \subseteq G$ and $x = (x_f)_{f\in F} \in \{1,\dots,N\}^F$, let $b(F,x) = \bigotimes_{f\in F} 1_{\left\{x_f\right\}} \otimes 1_{A^{G\setminus F}}$. As we have seen in Example 5.9, $\mathcal{B} = \{1_{A^G}\} \cup \left\{ b(F,x) \colon \emptyset \neq F \subseteq G \text{ finite}, x \in \{1,\dots,N\}^F \right\}$ is a $\mathbb{Z}$-basis for $C(A^G,\mathbb{Z})$.

Consider the subspace $C_0(A^G \setminus X,\mathbb{Z}) \subseteq C(A^G,\mathbb{Z})$ of functions vanishing on X. We claim the following: An element $\sum z_{F,x} b(F,x) \in C(A^G,\mathbb{Z})$ ($z_{F,x} \in \mathbb{Z}$) lies in $C_0(A^G \setminus X,\mathbb{Z})$ if and only if for every (F,x) with $z_{F,x} \neq 0$, there exists $i \in I$ and $g \in G$ with $F_i \subseteq gF$ and $\pi_{F_i}(g.x) = x_i$.

Clearly, if (F,x) satisfies this property, then $b(F,x)$ lies in $C_0(A^G \setminus X, \mathbb{Z})$. Conversely, suppose that $\sum z_{F,x} b(F,x)$ lies in $C_0(A^G \setminus X, \mathbb{Z})$ but there exists $(\tilde{F}, \tilde{x})$ with $z_{\tilde{F},\tilde{x}} \neq 0$ such that for all $i \in I$ and $g \in G$, $F_i \nsubseteq g\tilde{F}$ or $\pi_{F_i}(g.\tilde{x}) \neq x_i$. Among all the $(\tilde{F}, \tilde{x})$ with this property, choose a pair such that $\tilde{F}$ is minimal. Define $w \in A^G$ by setting $w_\gamma = \tilde{x}_\gamma$ if $\gamma \in \tilde{F}$ and $w_\gamma = 0$ otherwise. Then $w \in X$, and by our choice of w and $(\tilde{F}, \tilde{x})$, we have $b(\tilde{F}, \tilde{x})(w) = 1$ and $b(F', x')(w) = 0$ for all $(F', x') \neq (\tilde{F}, \tilde{x})$ with $z_{F',x'} \neq 0$. Hence $\left(\sum z_{F,x} b(F,x)\right)(w) = z_{\tilde{F},\tilde{x}}$, which contradicts that $\sum z_{F,x} b(F,x)$ vanishes on X. This shows that

$$\mathcal{B}_v := \{b(F,x) \colon \text{There is } i \in I \text{ and } g \in G \text{ with } F_i \subseteq gF \text{ and } \pi_{F_i}(g.x) = x_i\}$$

is a $\mathbb{Z}$-basis for $C_0(A^G \setminus X, \mathbb{Z})$.

The canonical homomorphisms give rise to the exact sequence $0 \to C_0(A^G \setminus X, \mathbb{Z}) \to C(A^G, \mathbb{Z}) \to C(X, \mathbb{Z}) \to 0$. One way to see this would be to apply K-theory to the exact sequence $0 \to C_0(A^G \setminus X) \to C(A^G) \to C(X) \to 0$. Therefore, the image $\mathcal{B}_X$ of $\mathcal{B} \setminus \mathcal{B}_v$ under the canonical projection $C(A^G, \mathbb{Z}) \twoheadrightarrow C(X, \mathbb{Z})$ is a $\mathbb{Z}$-basis for $C(X, \mathbb{Z})$. As $\mathcal{B}_v$ is clearly G-invariant, so is $\mathcal{B}_X$. Moreover, $1_X \in \mathcal{B}_X$, and G acts freely on $\mathcal{B}_X \setminus \{1_X\} \cong \mathcal{B} \setminus (\mathcal{B}_v \cup \{1_{A^G}\})$. Therefore, $G \curvearrowright X$ is $\mathbb{Z}G$-free.

6. Conclusions

Now we are ready for the proofs of Theorem 1.4 and Theorem 1.5.

Theorem (Theorem 1.4)**.** *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be topologically free systems. Assume that X is compact and that $C(X, \mathbb{Z}) = \mathbb{Z} \cdot 1 \oplus N$ as $\mathbb{Z}G$-modules, with $\mathrm{pd}_{\mathbb{Z}G}(N) < \mathrm{cd}(G) - 1$. Furthermore, let G be a duality group and H a solvable group. Then $G \curvearrowright X$ and $H \curvearrowright Y$ must be conjugate if they are continuously orbit equivalent.*

Proof of Theorem 1.4. Since G is a duality group, it is by definition (see [3, Chapter VIII, § 10]) of type FP and hence finitely generated by [3, Chapter VIII, § 5]. Hence, by Theorem 1.9 (proven in [18] and [20]), H is finitely generated, and G is quasi-isometric to H. Therefore, G is amenable (see [7, Chapter IV, 50. Geometric properties]). Moreover, Theorem 1.8 implies that $G \curvearrowright X$ is continuous H-cocycle rigid. Now assume $G \curvearrowright X \ \sim_{\mathrm{coe}} \ H \curvearrowright Y$. Since G is torsion-free and amenable, and because $G \curvearrowright X$ is continuous H-cocycle rigid, Theorem 1.7 implies that $G \curvearrowright X \ \sim_{\mathrm{conj}} \ H \curvearrowright Y$. □

Theorem (Theorem 1.5)**.** *The following topologically free systems are continuous orbit equivalence rigid:*

- *Topological Bernoulli actions $G \curvearrowright X_0^G$, where X_0 is a compact space with $|X_0| > 1$ and G is a solvable duality group;*
- *topologically free subshifts of $G \curvearrowright A^G$ whose forbidden words avoid a fixed letter, where $|A| < \infty$ and G is a solvable group.*

Proof of Theorem 1.5. Let $G \curvearrowright X$ be one of the systems in our list, and let $H \curvearrowright Y$ be an arbitrary topologically free system with $G \curvearrowright X \ \sim_{\mathrm{coe}} \ H \curvearrowright Y$. Since $G \curvearrowright X$ has a global fixed point, we must have $G \cong H$ by Corollary 2.12. If both G and H are trivial, there is nothing to show. If both G and H are isomorphic to $\mathbb{Z}$, our result follows from Theorem 3.2 as $G \curvearrowright X$ is topologically transitive. In the remaining case, we must have $\mathrm{cd}(G) > 1$, and then

$G \curvearrowright X \sim_{\text{conj}} H \curvearrowright Y$ follows from Theorem 1.4 and the observation that $G \curvearrowright X$ is $\mathbb{Z}G$-free (see Example 5.9 and Example 5.10). $\square$

Remark 6.1. If $G = \mathbb{Z}^d$, $d \geq 2$, as in the examples (golden mean, iceberg model, hard core model) mentioned in Theorem 1.5, the proof of Theorem 1.5 actually does not use non-abelian cohomology because it does not use Lemma 5.3 and Corollary 5.5.

Rigidity of the systems in Remark 1.6 follows from cocycle rigidity [26, § 10] and Theorem 1.7.

References

[1] S. Barlak and X. Li, *Cartan subalgebras and the UCT problem*, preprint, arXiv:1511.02697v2.

[2] M. Boyle and J. Tomiyama, *Bounded topological orbit equivalence and C*-algebras*, J. Math. Soc. Japan *50* (1998), 317–329.

[3] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics *87*, Springer, 1982.

[4] R. Burton and J.E. Steif, *Some 2-d Symbolic Dynamical Systems: Entropy and Mixing*, Ergodic Theory of $\mathbb{Z}^d$-actions (LMS Lecture Notes Series 228), Eds. M. Pollicott and K. Schmidt, Cambridge University Press, Cambridge, 1996, pp. 297–305.

[5] N.-P. Chung, Y. Jiang, *Continuous Cocycle Superrigidity for Shifts and Groups with One End*, preprint, arXiv:1603.00114.

[6] J. Cuntz, S. Echterhoff and X. Li, *On the K-theory of crossed products by automorphic semigroup actions*, Quart. J. Math. 64 (2013), 747–784.

[7] P. de la Harpe, *Topics in geometric group theory*, Chicago Lectures in Math., Univ. of Chic. Press, 2000.

[8] J. Feldman and C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. I*, Trans. Amer. Math. Soc. *234* (1977), 289–324.

[9] J. Feldman and C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras. II*, Trans. Amer. Math. Soc. *234*(1977), 325–359.

[10] A. Furman, *Orbit equivalence rigidity*, Ann. of Math. (2) *150* (1999), 1083–1108.

[11] D. Gaboriau, *Orbit equivalence and measured group theory*, in Proceedings of the ICM (Hyderabad, India, 2010), Vol. III, Hindustan Book Agency, 2010, pp. 1501–1527.

[12] T. Giordano, H. Matui, I.F. Putnam and C.F. Skau, *Orbit equivalence for Cantor minimal $\mathbb{Z}^2$-systems*, J. Amer. Math. Soc. *21* (2008), 863–892.

[13] T. Giordano, H. Matui, I.F. Putnam and C.F. Skau, *Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems*, Invent. Math. *179* (2010), 119–158.

[14] T. Giordano, I.F. Putnam and C.F. Skau, *Topological orbit equivalence and C*-crossed products*, J. Reine Angew. Math. *469* (1995), 51–111.

[15] A. Ioana, *W*-superrigidity for Bernoulli actions of property (T) groups*, J. Amer. Math. Soc. *24* (2011), 1175–1226.

[16] Y. Kida, *Measure equivalence rigidity of the mapping class group*, Ann. of Math. (2) *171* (2010), 1851–1901.

[17] X. Li, *Partial transformation groupoids attached to graphs and semigroups*, preprint, arXiv:1603.09165, accepted for publication in Int. Math. Res. Not.

[18] X. Li, *Quasi-isometry, Kakutani equivalence and applications to cohomology*, preprint, arXiv:1604.07375.

[19] K. Matsumoto and H. Matui, *Continuous orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras*, Kyoto J. Math. *54* (2014), 863–877.

[20] K. Medynets, R. Sauer and A. Thom, *Cantor systems and quasi-isometry of groups*, preprint, arXiv:1508.07578.

[21] N. Monod and Y. Shalom, *Orbit equivalence rigidity and bounded cohomology*, Ann. of Math. (2) *164* (2006), 825–878.

[22] S. Popa, *Cocycle and orbit equivalence superrigidity for malleable actions of w-rigid groups*, Invent. Math. *170* (2007), 243–295.

[23] S. Popa, *On the superrigidity of malleable actions with spectral gap*, J. Amer. Math. Soc. *21* (2008), 981–1000.

[24] J. Renault, *Cartan subalgebras in C*-algebras*, Irish Math. Soc. Bulletin *61* (2008), 29–63.
[25] K. Schmidt, *The cohomology of higher-dimensional shifts of finite type*, Pacific J. Math. *170* (1995), 237–270.
[26] K. Schmidt, *Tilings, fundamental cocycles and fundamental groups of symbolic* $\mathbb{Z}^d$*-actions*, Erg. Th. Dyn. Sys. *18* (1998), 1473–1525.
[27] J.P. Serre, *Local Fields*, Springer, Berlin Heidelberg New York, 1979.
[28] J.P. Serre, *Galois Cohomology*, Springer, Berlin Heidelberg New York, 1997.
[29] I.M. Singer, *Automorphisms of finite factors*, Amer. J. Math. *77* (1955), 117–133.
[30] Y. Suzuki, *Amenable minimal Cantor systems of free groups arising from diagonal actions*, preprint, arXiv: 1312.7098v4, accepted for publication in J. Reine Angew. Math.
[31] S. Vaes, *Rigidity results for Bernoulli actions and their von Neumann algebras* (after Sorin Popa), Séminaire Bourbaki, exp. no. 961, Astérisque *311* (2007), 237–294.
[32] S. Vaes, *Rigidity for von Neumann algebras and their invariants*, in Proceedings of the ICM (Hyderabad, India, 2010), Vol. III, Hindustan Book Agency, 2010, pp. 1624–1650.
[33] R.J. Zimmer, *Ergodic theory and semisimple groups*, Vol. 81 of Monogr. in Math., Birkhäuser, Basel, 1984.

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